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THE FIXED POINT METHOD FOR INTUITIONISTIC FUZZY STABILITY OF JENSEN-TYPE FUNCTIONAL EQUATION

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Abstract

Recently, the intuitionistic fuzzy stability of Jensen-type functional equation was proved. In this note, we show the intuitionistic fuzzy stability of the Jensen-type functional equation by using the fixed point alternative.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [14] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers Theorem was generalized by Aoki [1] for additive mappings, and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we call *generalized Hyers-Ulam-*

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Rassias stability of functional equations. In 1990, Rassias [12] asked whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [7] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [7] and Rassias and Semrl [11] that one cannot prove an analogous theorem when $p = 1$. In 1994, a generalization was obtained by Gavruta [8], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem.

In the following, we first recall some fundamental results in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$, if and only if $x = y$;

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

We recall the following theorem of Diaz and Margolis [6].

Theorem 1.1 (see [6]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty, \tag{1.1}$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J*;

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$*;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy),$ *for all* $y \in Y.$

In 2003, Cadariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation (see [3], [4]) for the first time. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

The new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t-representable was introduced by Shakeri [13]. We refer to [13] for the notions appeared below.

Consider the set L^* and the order relation $\leq_{L^*}$ defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \leq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [5].

A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-norm* if it satisfies the following conditions: (a) $*$ is associative and commutative; (b) $*$ is continuous; (c) $a * 1 = a$ for all $a \in [0, 1]$; (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

An intuitionistic fuzzy set $A_{\xi, \eta}$ in a universal set U is an object $A_{\xi, \eta} = \{(\xi_A(u), \eta_A(u)) : u \in U\}$, where for all $u \in U$, $\xi_A(u) \in [0, 1]$, and $\eta_A(u) \in [0, 1]$ are called the *membership degree* and the *non-membership degree*, respectively, of $u \in A_{\xi, \eta}$ and, furthermore, they satisfy $\xi_A(u) + \eta_A(u) \leq 1$.

A triangular norm (t-norm) on L^* is a mapping $T : (L^*)^2 \to L^*$ satisfying the following conditions: $\forall x, y, x', y', z \in L^*$, (a) $(T(x, 1_{L^*}) = x)$ (boundary condition); (b) $(T(x, y) = T(y, x))$ (commutativity); (c)

$(T(x, T(y, z)) = T(T(x, y), z))$ (associativity); (d) $(x \leq_{L^*} x'$ and $y \leq_{L^*} y'$ $\Rightarrow T(x, y) \leq_{L^*} T(x', y'))$ (monotonicity).

If $(L^*, \leq_{L^*}, T)$ is an Abelian topological monoid with unit 1_{L^*}, then T is said to be a *continuous t-norm*.

The definitions of an intuitionistic fuzzy normed space (briefly IFN-space) is given below (see [13]).

Definition 1.2. Let μ and υ be membership and non-membership degree of an intuitionistic fuzzy set from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + \upsilon_x(t) \leq 1$, for all $x \in X$ and $t > 0$. The triple $(X, P_{\mu, \upsilon}, T)$ is said to be an *intuitionistic fuzzy normed space* (briefly *IFN-space*) if X is a vector space, T is a continuous t-representable, and $P_{\mu, \upsilon}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(a) $P_{\mu, \upsilon}(x, 0) = 0_{L^*}$;

(b) $P_{\mu, \upsilon}(x, t) = 1_{L^*}$, if and only if $x = 0$;

(c) $P_{\mu, \upsilon}(\alpha x, t) = P_{\mu, \upsilon}(x, \frac{t}{|\alpha|})$, for all $\alpha \neq 0$;

(d) $P_{\mu, \upsilon}(x + y, t + s) \geq T(P_{\mu, \upsilon}(x, t), P_{\mu, \upsilon}(y, s))$.

In this case, $P_{\mu, \upsilon}$ is called an *intuitionistic fuzzy norm*. Here, $P_{\mu, \upsilon}(x, t) = (\mu_x(t), \upsilon_x(t))$.

In this short note, we show the intuitionistic fuzzy stability of the generalized Jensen-type functional equation

$$2f(x + y) - 2f(x - y) = 2f(y)$$

by using the fixed point alternative.

2. Main Results

In the following, we show the intuitionistic fuzzy stability of the Jensen-type functional equation by using the fixed point alternative. A direct proof was given by Shakeri [13].

Theorem 2.1. *Let X be a linear space,* $(Z, P'_{\mu,\upsilon}, M)$ *be an IFN-space,* $\phi : X \times X \to Z$ *be a function such that for some* $0 < \alpha < 2$,

$$P'_{\mu,\upsilon}(\phi(2x, 2x), t) \geq_{L^*} P'_{\mu,\upsilon}(\alpha\phi(x, x), t)\ (x \in X, t > 0), \tag{2.1}$$

and

$$\lim_{n\to\infty} P'_{\mu,\upsilon}(\phi(2^n x, 2^n y), 2^n t) = 1_{L^*}, \tag{2.2}$$

for all $x, y \in X$ *and* $t > 0$. *Let* $(Y, P_{\mu,\upsilon}, M)$ *be a complete IFN-space. If* $f : X \to Y$ *is a mapping such that* $\forall x, y \in X, t > 0$,

$$P_{\mu,\upsilon}(f(x+y) - f(x-y) - 2f(y), t) \geq_{L^*} P'_{\mu,\upsilon}(\phi(x, y), t), \tag{2.3}$$

and $f(0) = 0$, *then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$P_{\mu,\upsilon}(f(x) - A(x), t) \geq_{L^*} P'_{\mu,\upsilon}(\phi(x, y), (2-\alpha)t). \tag{2.4}$$

Proof. Put $y = x$ in (2.3), we have

$$P_{\mu,\upsilon}\left(\frac{f(2x)}{2} - f(x), t\right) \geq_{L^*} P'_{\mu,\upsilon}(\phi(x, x), 2t),$$

for all $x \in X$ and $t > 0$.

Consider the set $E = \{g : X \to Y\}$ and define a generalized metric d on E by

$$d(g, h) = \inf\{c \in R^+ : P_{\mu,\upsilon}(g(x) - h(x), ct) \geq_{L^*} P'_{\mu,\upsilon}(\phi(x, x), t), \forall x \in X, t > 0\}.$$

It is easy to show that (E, d) is complete. Define $J : E \to E$ by $Jg(x) = \frac{1}{2} g(2x)$ for all $x \in X$. It is not difficult to see that

$$d(Jg,\ Jh) \leq \frac{\alpha}{2} d(g,\ h),$$

for all $g,\ h \in E$. It follows from above that

$$d(f,\ Jf) \leq \frac{1}{2}.$$

It follows from Theorem 1.1 that J has a fixed point in the set $E_1 = \{h \in E : d(f,\ h) < \infty\}$. Let A be the fixed point of J. It follows from $\lim_n d(J^n f,\ A) = 0$ that

$$A(x) = \lim_n \frac{1}{2^n} f(2^n x),$$

for all $x \in X$. Since $d(f,\ A) \leq \dfrac{1}{2 - \alpha}$,

$$P_{\mu, v}(f(x) - A(x),\ t) \geq_{L^*} P'_{\mu, v}(\phi(x,\ y),\ (2 - \alpha)t).$$

It follows from (2.3) that we have

$$P_{\mu,v}\left(\frac{1}{2^n}[f(2^n(x+y)) - f(2^n(x-y)) - 2f(2^n y)],\ t\right) \geq_{L^*} P'_{\mu,v}(\phi(2^n x,\ 2^n y),\ 2^n t).$$

By (2.2), A is additive.

The uniqueness of A follows from the fact that A is the unique fixed point of J with the property that

$$P_{\mu, v}(f(x) - A(x),\ t) \geq_{L^*} P'_{\mu, v}(\phi(x,\ y),\ (2 - \alpha)t).$$

This completes the proof. □

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